KOVALEVSKAYA TOP AND GENERALIZATIONS OF INTEGRABLE SYSTEMS[1]

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Abstract

Generalizations of the Kovalevskaya, Chaplygin, Goryachev–Chaplygin and Bogoyavlensky systems on a bundle are considered in this paper. Moreover, a method of introduction of separating variables and action–angle variables is described. Another integration method for the Kovalevskaya top on the bundle is found. This method uses a coordinate transformation that reduces the Kovalevskaya system to the Neumann system. The Kolosov analogy is considered. A generalization of a recent Gaffet system to the bundle of Poisson brackets is obtained at the end of the paper.

Dedicated to the 150-th anniversary of S. V. Kovalevskaya

It was the 150-th anniversary on January 15, 2000 of Sofia Vasilievna Kovalevskaya, an outstanding women-mathematician, whose works made a great contribution into development of the world science.

Her literature works also were of great importance for the Russian culture where she developed progressive and even revolutionary ideas.

[1]REGULAR AND CHAOTIC DYNAMICS V.6, No. 1, 2001
Received December 12, 2000
AMS MSC 70E17, 70E40

The most important results were obtained by S. V. Kovalevskaya in the theory of differential equations and in rigid body dynamics. S. V. Kovalevskaya stated and proved the existence and uniqueness theorem for solutions of partial differential equations [3] (Cauchy-Kovalevskaya theorem). She also found a new method of integrability analysis (the so-called Painlevé–Kovalevskaya test). The substance of this test is as follows.

Let a system of differential equations

$$z_i = f_i(z_1, \ldots, z_n), \quad 1 \leq i \leq n \tag{1}$$

be invariant with respect to the similarity transformations

$$t \to t/\alpha, \quad z_i \to \alpha^{g_i} z_i, \quad 1 \leq i \leq n.$$

Let us construct a matrix (the Kovalevskaya matrix)

$$K_{ij} = \frac{\partial f_i}{\partial z_j}(c) + g_j \delta_{ij},$$

where the constants c_i satisfy the algebraic system of equations $f_i(c_1, \ldots, c_n) = -g_i c_i$ $(1 \leq i \leq n)$. Kovalevskaya required the conditions of integrality and non-negativity of eigenvalues of K for the uniqueness of a general solution of (1) on a complex time plane. Lyapunov [27] continued these investigations of Kovalevskaya and related the meromorphy of a general solution with properties of the variational equations system. The Lyapunov method was further developed by S. L. Ziglin, who proved strict results concerning nonexistence of additional single-valued integrals.

The works on rotation of a heavy rigid body about a fixed point are the most famous [2] (1888). In these works a new integrable case of the Euler–Poisson equations was found and its solution in quadratures was obtained. The French Academy of Sciences highly appreciated this S. V. Kovalevskaya result and awarded her with the Prix Bordin in 1888. Let us note that the Academy of Sciences had twice announced the prize contest on investigations on this subject, however the prize was not awarded. S. V. Kovalevskaya was awarded with a prize of the Sweden Academy of Sciences for the second memoir on the problem of rotation of a rigid body in spring 1889. A lot of questions concerning the motion of the Kovalevskaya top are still open.

All integrable cases under consideration are well known in rigid body dynamics. However, not for all of them the generalizations from the algebra $e(3)$ to the bundle of brackets $\mathcal{L}_x$

$$\{M_i, M_j\} = -\varepsilon_{ijk} M_k, \quad \{M_i, \gamma_j\} = -\varepsilon_{ijk} \gamma_k, \quad \{\gamma_i, \gamma_j\} = -x \varepsilon_{ijk} M_k, \tag{2}$$

have been investigated yet. Let us note that $\mathcal{L}_x$ possesses two quadratic Casimir functions

$$\begin{aligned} l &= (\boldsymbol{M}, \boldsymbol{\gamma}), \\ c &= x(\boldsymbol{M}, \boldsymbol{M}) + (\boldsymbol{\gamma}, \boldsymbol{\gamma}). \end{aligned} \tag{3}$$

These problems naturally appear for a system which describes dynamics of a rigid body in an ideal fluid, connected bodies, motion of a body with elliptic hollow filled in with a liquid, etc. Generalizations of known integrable cases to the algebra $so(4)$ and separating variables for several of them can be found in [12].

1 The Kovalevskaya Top and its Generalizations

1.1 Integrals of the Kovalevskaya Case and its Generalizations

S. V. Kovalevskaya found a new general integrable case of the Euler-Poisson equations in 1888 [2] basing on pure mathematical considerations. She was developing ideas of K. Weierstrass, P. Painlevé and H. Poincaré concern the investigation of analytical extension of solutions of a system of ordinary differential equations to the complex time plane. Kovalevskaya assumed that a general solution does not possess other singularities except for poles in the complex plane in integrable cases. That enabled to find existence conditions for the additional integral. Moreover, S. V. Kovalevskaya found a quite nonobvious system of variables. The motion equations expressed in these variables are of the Abel-Jacobi type. She also obtained an explicit solution in theta-functions.

In the Kovalevskaya case the body is symmetric: $I_1 = I_2 = 2I_3$, where I_i $(i = 1, 2, 3)$ are principal central moments of inertia, the centre of mass is situated in the equatorial plane of the inertia ellipsoid. The Hamiltonian has the form

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2) + \gamma_1. \tag{4}$$

The additional integral found by Kovalevskaya can be represented as follows:

$$F_3 = \left(\frac{M_1^2 - M_2^2}{2} - \boldsymbol{\gamma}_1\right)^2 + (M_1 M_2 - \boldsymbol{\gamma}_2)^2 \tag{5}$$

(without loss of generality it is assumed that the radius-vector of the centre of mass is $\boldsymbol{r} = (1, 0, 0)$, weight equals unit). Let us note that the integral F_3 does not have an evident symmetry origin.

The Kovalevskaya case (on $e(3)$) can be generalized to the bundle of brackets $\mathcal{L}_x$ (1).

Thus, the additional integral (the Kovalevskaya integral) will have the form

$$\begin{gathered} k^2 = k_1^2 + k_2^2, \\ k_1 = M_1^2 - M_2^2 - 2\gamma_1 + x, \quad k_2 = 2M_1M_2 - 2\gamma_2. \end{gathered} \tag{6}$$

Generalizations of integrable cases can be associated with introduction of preserving integrability additional terms in the Hamiltonian without the change of structure of the Poisson brackets algebra (for example, introduction of a gyrostat or singular terms). Generalizations can be also associated with the change of the algebra structure without change of the Hamiltonian (for example, generalization to the bundle of brackets $\mathcal{L}_x$), with a simultaneous change of the Hamiltonian and Poisson brackets, with a generalization to the case of several force fields.

The motion equations can be generalized if one introduces a constant gyrostatic moment induced by a balanced rotor which is fixed in a rigid body and rotates with a constant angular velocity. An analogous moment occurs in motion of a rigid body with multiply connected (that admits a possibility of appearance of nonzero circulation) hollows containing an ideal incompressible liquid [18]. In this case the algebra and motion equations do not change. However, a linear in momenta term appears in the Hamiltonian.

A gyrostat in the Kovalevskaya top was introduced by Yehia (1987) and Komarov (1987):

$$\begin{aligned} H =& \frac{1}{2}\left(M_1^2 + M_2^2 + 2\left(M_3 - \frac{\lambda}{2}\right)^2\right) + r_1\gamma_1, \\ F =& (M_1^2 - M_2^2 - 2r_1\gamma_1)^2 + (2M_1M_2 - 2r_1\gamma_2)^2 + \\ &+ 4\lambda(M_3 - \lambda)(M_1^2 + M_2^2) - 8r_1\lambda M_1\gamma_3. \end{aligned}$$

In [22] D. N. Goryachev suggested a generalization of the Kovalevskaya case at the zero area constant $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$

$$\begin{aligned} H &= \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2) + \frac{a}{\gamma_3^2} + 2b_1\gamma_1\gamma_2 + b_2(\gamma_2^2 - \gamma_1^2) + r_1\gamma_1 + r_2\gamma_2, \\ F &= 4\left(M_1M_2 - 2a\frac{\gamma_1\gamma_2}{\gamma_3^2} + b_1\gamma_3^2 - r_1\gamma_2 - r_2\gamma_1\right)^2 + \\ &\quad + \left(M_1^2 - M_2^2 - 2a\frac{(\gamma_1^2 - \gamma_2^2)}{\gamma_3^2} - 2b_2\gamma_3^2 - 2r_1\gamma_1 + 2r_2\gamma_2\right)^2, \\ &\qquad a,\ b_1,\ b_2,\ r_1,\ r_2 = const. \end{aligned} \tag{7}$$

A physical sense of the singular term can be interpreted in quantum mechanics and with the help of reduction with symmetry suggested in [23] for a variant of rigid body motion in superposition of two homogeneous fields. Let us note that at $a = 0$ the integrable generalization under consideration was indicated by S. A. Chaplygin

in his work [24], dedicated to a new integrable case of the Kirchhoff equations. D. N. Goryachev did not refer to [24], although his work [22] appeared later.

In the Yehia work [25] a constant gyrostatic moment along the axis of dynamical symmetry is added in the Hamiltonian (7) at the zero value of the area integral

$$
\begin{aligned}
H &= \frac{1}{2}(M_1^2 + M_2^2 + 2(M_3+\lambda)^2) + \frac{a}{\gamma_3^2} + 2b_1\gamma_1\gamma_2 + b_2(\gamma_2^2 - \gamma_1^2) + r_1\gamma_1 + r_2\gamma_2, \\
F &= 4\left(M_1M_2 - 2a\frac{\gamma_1\gamma_2}{\gamma_3^2} + b_1\gamma_3^2 - r_1\gamma_2 - r_2\gamma_1\right)^2 + \\
&\quad + \left(M_1^2 - M_2^2 - 2a\frac{(\gamma_1^2-\gamma_2^2)}{\gamma_3^2} - 2b_2\gamma_3^2 - 2r_1\gamma_1 + 2r_2\gamma_2\right)^2 - \\
&\quad - 8\lambda(M_3+2\lambda)(M_1^2+M_2^2) - 16a\lambda(M_3+2\lambda)\left(1+\frac{1}{\gamma_3^2}\right) + \\
&\quad + 16\lambda\gamma_3\Big(M_1(r_1 + b_1\gamma_2 - b_2\gamma_1) + M_2(r_2 + b_1\gamma_1 + b_2\gamma_2)\Big).
\end{aligned}
$$

$$a,\ b_1,\ b_2,\ r_1,\ r_2,\ \lambda = const.$$

In [26] Yehia suggested another generalization of the Kovalevskaya case with the help of the introduction of a singular (but another) term, which has not obtained a clear physical interpretation yet

$$
\begin{aligned}
H &= \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2) + r_1\gamma_1 + r_2\gamma_2 + \frac{\varepsilon}{\sqrt{\gamma_1^2+\gamma_2^2}}, \\
F &= (M_1^2 - M_2^2 - 2r_1\gamma_1 + 2r_2\gamma_2)^2 + 4(M_1M_2 - r_1\gamma_2 - r_2\gamma_1)^2 + +4\frac{\varepsilon(M_1^2+M_2^2)}{\sqrt{\gamma_1^2+\gamma_2^2}} + 4\frac{\varepsilon^2}{\gamma_1^2+\gamma_2^2},
\end{aligned}
$$

$$\varepsilon,\ r_1,\ r_2 = const.$$

This integral curiously gives a general integrable case.

Another generalization of the Kovalevskaya case is associated with introduction of two different singular potentials in the Hamiltonian [25]:

$$H = \tfrac{1}{2}(M_1^2 + M_2^2 + 2M_3^2) + r_1\gamma_1 + r_2\gamma_2 + \tfrac{\varepsilon}{\sqrt{\gamma_1^2+\gamma_2^2}} + \tfrac{a}{\gamma_3^2},$$

$$a,\ \varepsilon,\ r_1,\ r_2 = const.$$

A particular integral at the zero area constant has the form:

$$
\begin{aligned}
F = &\left(M_1^2 - M_2^2 - 2r_1\gamma_1 + 2r_2\gamma_2 - \frac{2a(\gamma_1^2-\gamma_2^2)}{\gamma_3^2}\right)^2 + \\
&+ 4\left(M_1M_2 - r_1\gamma_2 - r_2\gamma_1 - \frac{2a\gamma_1\gamma_2}{\gamma_3^2}\right)^2 + \\
&+ 4\varepsilon\left(\frac{M_1^2+M_2^2}{\sqrt{\gamma_1^2+\gamma_2^2}} + \frac{\varepsilon}{\gamma_1^2+\gamma_2^2} + \frac{2a\sqrt{\gamma_1^2+\gamma_2^2}}{\gamma_3^2}\right).
\end{aligned}
$$

Let us note that the integral F is invalid in [25], though it is probably connected with a strange misprint.

There exist generalizations of the Kovalevskaya and Chaplygin integrable cases which include gyrostatic terms on the entire bundle $\mathcal{L}_x$. The analogue of the area constant is also supposed to be zero in this case

$$(\boldsymbol{M},\, \boldsymbol{\gamma}) = 0.$$

It is more convenient to represent the Hamiltonian as follows:

$$\begin{gathered} H = \frac{1}{2}\Big((1 + xa_1)M_1^2 + (1 + xa_2)M_2^2 + 2\Big(M_3 - \frac{\lambda}{2}\Big)^2\Big) + \\ + a_2\gamma_1^2 + a_1\gamma_2^2 + \frac{1}{2}(a_1 + a_2)\gamma_3^2 + r_1\gamma_1 + r_2\gamma_2. \end{gathered} \tag{8}$$

This form of the Hamiltonian at $\lambda = 0$ differs from the representation indicated in [12] on the Casimir function $\frac{1}{2}(a_1 + a_2)(x\boldsymbol{M}^2 + \boldsymbol{\gamma}^2)$. The result regarding integrability of the system for $\lambda \neq 0$ is new.

In this case the integral has the form

$$\begin{gathered} K = k_1^2 + \alpha_1\alpha_2 k_1^2 - \lambda\{k_1,\, k_2\} - 4\lambda^2(M_1^2 + M_2^2), \\ k_1 = \alpha_1 M_1^2 - \alpha_2 M_2^2 - (a_1 - a_2)\gamma_3^2 - 2(\gamma_1 r_1 + \gamma_2 r_2) + x\frac{\alpha_1 r_1^2 - \alpha_2 r_2^2}{\alpha_1\alpha_2}, \\ k_2 = M_1 M_2 - 2\frac{\alpha_1 r_1 \gamma_2 + \alpha_2 r_2 \gamma_1}{\alpha_1\alpha_2} + 2x\frac{r_1 r_2}{\alpha_1\alpha_2}, \\ \{k_1,\, k_2\} = 4M_3\Big(\alpha_1 M_1^2 - \alpha_2 M_2^2 - x\frac{\alpha_1 r_1^2 + \alpha_2 r_2^2}{\alpha_1\alpha_2}\Big) - 4\gamma_3((a_1 - a_2)(M_1\gamma_1 - M_2\gamma_2) - \\ -2(M_1 r_1 + M_2 r_2)). \end{gathered} \tag{9}$$

Remark. *The motion equations for K can be represented in the form*

$$\dot{K} = \{K,\, H\} = (a_1 - a_2)(\boldsymbol{M},\, \boldsymbol{\gamma})F(\boldsymbol{M},\, \boldsymbol{\gamma}).$$

Thus, this case becomes a general integrable case at $a_1 = a_2$ that corresponds to the gyrostatic generalization of the Kovalevskaya case.

1.2 Separation of Variables on the Bundle $\mathcal{L}_x$

Our reduction to the Abel equations on the bundle of brackets $\mathcal{L}_x$ uses arguments of G. K. Suslov, who suggested a new method of integration of the Kovalevskaya case in his famous treatise [4].

Let us set new variables

$$
\begin{aligned}
z_1 &= M_1 + iM_2, & z_2 &= M_1 - iM_2, \\
\zeta_1 &= k_1 + ik_2, & \zeta_2 &= k_1 - ik_2,
\end{aligned}
$$

$$
\zeta_1\zeta_2 = k^2.
$$

Let us take advantage of the motion equations for z_1, z_2

$$
i\dot{z}_1 = M_3 z_1 - \gamma_3, \quad -i\dot{z}_2 = M_3 z_2 - \gamma_3
$$

and formulate $\gamma_1, \gamma_2, \gamma_3, M_3$ as

$$
\begin{aligned}
\gamma_1 &= \frac{1}{4}\left(z_1^2 + z_2^2\right) - \frac{1}{4}\left(\zeta_1 + \zeta_2\right) + \frac{x}{2}, \\
\gamma_2 &= -\frac{i}{4}\left(z_1^2 - z_2^2\right) + \frac{i}{4}\left(\zeta_1 - \zeta_2\right), \\
\gamma_3 &= i\frac{\dot{z}_1 z_2 + \dot{z}_2 z_1}{z_1 - z_2}, \quad M_3 = i\frac{\dot{z}_1 + \dot{z}_2}{z_1 - z_2}.
\end{aligned} \tag{10}
$$

Then insert the obtained expressions into the integrals (2) and the Hamiltonian (4). Solving the obtained equations, one finds

$$
\begin{aligned}
\dot{z}_1^2 &= R_1 - \frac{\zeta_1}{4}\left(z_1 - z_2\right)^2, \\
\dot{z}_2^2 &= R_2 - \frac{\zeta_2}{4}\left(z_1 - z_2\right)^2, \\
\dot{z}_1\dot{z}_2 &= -R - \frac{1}{4}\left(2h - x\right)\left(z_1 - z_2\right)^2,
\end{aligned} \tag{11}
$$

where

$$
\begin{gathered}
R = R(z_1, z_2) = \tfrac{1}{4} z_1^2 z_2^2 - \tfrac{h}{2}\left(z_1^2 + z_2^2\right) + l\left(z_1 + z_2\right) + \tfrac{k^2}{4} - c + xh - \tfrac{x^2}{4}, \\
R_1 = R(z_1, z_1), \qquad R_2 = R(z_2, z_2),
\end{gathered} \tag{12}
$$

here h is a constant of the energy integral (4).

We still need to exclude ζ_1, ζ_2 from the obtained equations with the help of the Kovalevskaya integral (6)

$$
\left(R_1 - \dot{z}_1^2\right)\left(R_2 - \dot{z}_2^2\right) = \frac{\zeta_1\zeta_2}{16}(z_1 - z_2)^4 = \frac{k^2}{16}(z_1 - z_2)^4.
$$

Rearrange terms in the obtained relation and represent the latter in the form

$$\begin{aligned} \left(\tfrac{\dot z_1}{\sqrt{R_1}} + \tfrac{\dot z_2}{\sqrt{R_2}}\right)^2 &= \left(\tfrac{\dot z_1 \dot z_2}{\sqrt{R_1 R_2}} + 1\right)^2 - \tfrac{k^2(z_1-z_2)^4}{16 R_1 R_2} = f_1 \\ \left(\tfrac{\dot z_1}{\sqrt{R_1}} - \tfrac{\dot z_2}{\sqrt{R_2}}\right)^2 &= \left(\tfrac{\dot z_1 \dot z_2}{\sqrt{R_1 R_2}} - 1\right)^2 - \tfrac{k^2(z_1-z_2)^4}{16 R_1 R_2} = f_2, \end{aligned} \tag{13}$$

where $\dot z_1 \dot z_2$ are substituted from (11).

The last step is the introduction of the Kovalevskaya variables by formulae

$$s_1 = \frac{R - \sqrt{R_1 R_2}}{2(z_1 - z_2)^2}, \qquad s_2 = \frac{R + \sqrt{R_1 R_2}}{2(z_1 - z_2)^2}. \tag{14}$$

Let us express $R = (s_1 + s_2)(z_1 - z_2)^2$, $\sqrt{R_1 R_2} = (s_2 - s_1)(z_1 - z_2)^2$ from these relations and insert them in the right-hand sides of (13). We obtain

$$\begin{gathered} f_1 = \tfrac{f(s_1)}{(s_1 - s_2)^2}, \quad f_2 = \tfrac{f(s_2)}{(s_1 - s_2)^2}, \\ f(s) = \left(2s + \tfrac{1}{4}(2h - x) + \tfrac{1}{4}k\right)\left(2s + \tfrac{1}{4}(2h - x) - \tfrac{1}{4}k\right). \end{gathered} \tag{15}$$

The relations for the left-hand sides of (13) define a transition to curvilinear coordinates

$$\frac{dz_1}{\sqrt{R_1}} + \frac{dz_2}{\sqrt{R_2}} = \frac{ds_1}{\sqrt{\varphi(s_1)}}, \qquad \frac{dz_1}{\sqrt{R_1}} - \frac{dz_2}{\sqrt{R_2}} = \frac{ds_2}{\sqrt{\varphi(s_2)}}, \tag{16}$$

where $\varphi(s)$ is a polynomial of the third degree,

$$\varphi(s) = 4s^3 + 2hs^2 + \left(\frac{1}{16}(2h - x)^2 - \frac{k^2}{16} + \frac{c}{4}\right) s + \frac{l^2}{16}.$$

Inserting (16) into (13), we obtain the motion equations in the variables s_1, s_2

$$\dot s_1 = \frac{\sqrt{f(s_1)\varphi(s_1)}}{s_1 - s_2}, \quad \dot s_2 = \frac{\sqrt{f(s_2)\varphi(s_2)}}{s_2 - s_1}. \tag{17}$$

The polynomial $\varphi(s)$ can be obtained with the help of standard methods using reduction of elliptic integrals to the standard from. Let us describe the Suslov method [4] here.

The Kovalevskaya variables are solutions of the quadratic equation

$$Q(z_1, z_2, s) = (z_1 - z_2)^2 s^2 - Rs + G = 0, \tag{18}$$

where

$$G = \frac{R^2 - R_1 R_2}{4(z_1 - z_2)^2} = -\frac{l}{8} z_1 z_2 (z_1 + z_2) + \frac{1}{64}\left((2h - x)^2 - k^2 + 4c\right)(z_1 + z_2)^2 - \frac{lh}{4}(z_1 + z_2) + \frac{l^2}{4}.$$

Let us calculate quadrates of derivatives $\left(\frac{\partial Q}{\partial s}\right)^2, \left(\frac{\partial Q}{\partial z_1}\right)^2, \left(\frac{\partial Q}{\partial z_2}\right)^2$ and using (18) exclude s, z_1, z_2 from them respectively. By a direct calculation we verify that

$$\left(\frac{\partial Q}{\partial s}\right)^2 = R_1 R_2, \quad \left(\frac{\partial Q}{\partial z_1}\right)^2 = \varphi(s) R_2, \quad \left(\frac{\partial Q}{\partial z_2}\right)^2 = \varphi(s) R_1.$$

Let us construct an exact differential of Q

$$dQ = \frac{\partial Q}{\partial z_1} dz_1 + \frac{\partial Q}{\partial z_2} dz_2 + \frac{\partial Q}{\partial s} ds = 0.$$

Dividing by $\sqrt{\varphi(s) R_1 R_2}$ and taking into account a possibility of rooting with different signs, we obtain the relation (16).

1.3 The Haine-Horozov Transformation for the Kovalevskaya System

Let us consider another method of integration of the Kovalevskaya system. This method uses a coordinate transformation analogous to the one indicated in [5]. The Kovalevskaya system reduces to the Neumann problem on a motion of a point on S^2 with the help of this transformation. That allows to write down a Lax pair for the Kovalevskaya top and implies a relation between integrals of both systems.

Let us denote the motion integrals of the Kovalevskaya system on $\mathcal{L}_x$ as follows:

$$\begin{gathered} 2H = \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2) + 2\gamma_1 = h, \\ c_1 = (\boldsymbol{M}, \boldsymbol{\gamma}), \quad c_2 = x(\boldsymbol{M}, \boldsymbol{M}) + (\boldsymbol{\gamma}, \boldsymbol{\gamma}) - k^2, \\ k^2 = \left(\frac{M_1^2 - M_2^2}{4} - \gamma_1 + x\right)^2 + \left(\frac{M_1 M_2}{2} - \gamma_2\right)^2 \end{gathered} \tag{19}$$

and transform variables by formulae

$$\begin{array}{ll} p_1 = \frac{M_1^2 + M_2^2 + 4}{4M_2}, & l_1 = \frac{4M_1\gamma_3 - M_3(M_1^2 + M_2^2 - 4)}{8M_2}, \\ p_2 = -i\frac{M_1}{M_2}, & l_2 = -i\frac{\gamma_3}{M_2}, \\ p_3 = i\frac{M_1^2 + M_2^2 - 4}{4M_2}, & l_3 = i\frac{4M_1\gamma_3 - M_3(M_1^2 + M_2^2 + 4)}{8M_2}. \end{array} \tag{20}$$

Then after the coordinate transformation (20) the motion equations can be written as

$$\dot{\boldsymbol{p}} = \boldsymbol{l} \times \boldsymbol{p}, \quad \dot{\boldsymbol{l}} = \mathbf{Q}\boldsymbol{p} \times \boldsymbol{p} + \frac{x(x-h)}{4}(p_1 + ip_3)\boldsymbol{y}, \tag{21}$$

where the matrix $\mathbf{Q}$ has the form

$$\mathbf{Q} = \frac{1}{4}\begin{pmatrix} c_2 - 1 & -ic_1 & i(1+c_2) \\ -ic_1 & -h & c_1 \\ i(1+c_2) & c_1 & 1 - c_2 \end{pmatrix}, \tag{22}$$

h, c_1, c_2, k^2 are determined by (19), the vector $\boldsymbol{y} = (-ip_2,\, i(p_1 + ip_3),\, p_2)$.

The motion equations (21) can be regarded as the system on $e(3)$ with the Hamiltonian

$$(\boldsymbol{l},\, \boldsymbol{l}) + (\mathbf{Q}\boldsymbol{p},\, \boldsymbol{p}) + \frac{x(x-h)}{4}(p_1 + ip_3)^2 - x = -\frac{h}{4},$$

i. e. as the Neumann system. Thus, the Kovalevskaya system on the bundle (1) can be integrated by reduction to the Neumann system. The integration of the latter is performed by standard methods.

Let us note that the Kovalevskaya system can not be reduced to the Neumann system with the use of the transformation (20) after introduction of a gyrostat.

1.4 A Generalization of the Kolosoff Analogy

G. V. Kolosoff considered a transformation in [7] which involves coordinates and time and reduces the Kovalevskaya problem on $e(3)$ to dynamics of a point on the Euclidean plane in $\mathbb{R}^2$ in a potential field, for which variables can be separated. This transformation is used to introduce action-angle variables in the Kovalevskaya problem in [8].

Let us consider an analogous procedure for the Kovalevskaya problem on the bundle (1). In this case an analog of the Kolosoff transformation gives dynamics of a particle on an axisymmetric nonconstant curvature surface.

Following [8], let us express the Hamiltonian from the motion equations. Let us change coordinates in (17)

$$s_i \to s_i - \frac{1}{2}(h + \frac{x}{4}), \quad i = 1,\, 2 \tag{23}$$

and represent them in the form

$$\frac{(s_1 - s_2)^2 \dot{s}_i^2}{f(s_i)} = g(s_i), \quad i = 1,\, 2$$

$$\begin{aligned} f(s) &= 4(s - \frac{x}{4} + \frac{1}{2}k)(s - \frac{x}{4} - \frac{1}{2}k), \\ g(s) &= 4s^3 - (2h + \frac{3}{2}x)s^2 + (c - k^2 + \frac{x^2}{4})s + \varkappa, \\ \varkappa &= \frac{1}{2}(k^2 h + 2l^2 - ch) + \frac{x}{4}(h^2 + k^2 - c) - \left(\frac{x}{4}\right)^3. \end{aligned} \tag{24}$$

Remark. *The form of the coordinate change (23) is dictated by the requirement that the energy constant in $g(s)$ was contained in even powers of s only.*

Substraiting the first equation from the second in (24), we eliminate the constant $\varkappa$ and express the energy from the obtained relation

$$\begin{aligned} H &= \frac{(s_1 - s_2)^2}{2(s_1^2 - s_2^2)}\Big(\frac{\dot{s}_1^2}{f(s_1)} - \frac{\dot{s}_2^2}{f(s_2)}\Big) + U(s_1,\, s_2), \\ U(s_1,\, s_2) &= \frac{2(s_1^2 + s_1 s_2 + s_2^2) + \frac{1}{2}(c - k^2 + \frac{x^2}{4})}{s_1 + s_2}. \end{aligned} \tag{25}$$

After the change of time $d\tau = 2(s_1 + s_2)\, dt$ and transition to the canonical momenta $p_i = \frac{\partial H}{\partial s_i'}$, $s_i' = \frac{ds_i}{d\tau}$, $i = 1,\, 2$, one obtains a system with separable variables.

Let us consider the variables $s_1 - \frac{x}{4}$, $s_2 - \frac{x}{4}$ as elliptic coordinates on a plane with the following relation with the Cartesian coordinates $u,\, v$

$$\begin{aligned} u &= \frac{2}{k}\Big(s_1 - \frac{x}{4}\Big)\Big(s_2 - \frac{x}{4}\Big) + \frac{k}{2}, \\ v &= \pm\frac{2}{k}\sqrt{\Big(s_1^2 - \frac{k^2}{4}\Big)\Big(\frac{k^2}{4} - s_2^2\Big)}. \end{aligned}$$

We obtain a system on a plane with the energy determined by the relations

$$\begin{aligned} H =& T + U, \\ T =& \frac{1}{2}\Big(1 + \frac{x}{2\rho}\Big)(u'^2 + v'^2), \\ U =& \frac{4\rho^2 - 2uk + c + 3x\rho + x^2}{2\rho + x} = \frac{2(\rho^2 + \rho_1^2) - k^2 + c + 3x\rho + x^2}{2\rho + x}, \\ \rho =& \sqrt{u^2 + v^2}, \quad \rho_1 = \sqrt{(u-k)^2 + v^2}. \end{aligned} \tag{26}$$

The system (26) describes the motion of a point along a curvilinear surface in a potential field. The Gauss curvature can be evaluated by the metrics which is defined by the kinetic energy T

$$K = -\frac{4x}{(2\rho + x)^3}.$$

1.5 The Action-Angle Variables for the Kovalevskaya Top on the Bundle

The method of introduction of the action-angle variables developed in this paper is analogous to the method considered in [8]. In comparison with a usually cited procedure due to A. P. Veselov and S. P. Novikov, this algorithm is more natural and uses an ordinary method of introduction of action-angle variables for the systems with separating variables.

Our algorithm consists of the following steps:

1. Construction of the Abel variables s_1, s_2, that commute in the original Poisson structure: $\{s_1,\, s_2\} = 0$.

Remark 1. *The existence of commuting set of the Abel variables s_1, s_2 can be established with the help of arguments due to [17]. These arguments are connected with the reduction of the equations (17) to the standard form on a torus.*

2. Introduction of the canonical momenta p_i with the help of the energy equation in the variables s_i, $\dot{s}_i$. These momenta must satisfy an additional requirement that the system p_i, s_i should be separable.

3. Having a set of separated variables, the action-angle variables can be introduced in accordance with a well known algorithm.

It is easy to verify that the Kovalevskaya variables (14) s_1, s_2 commute. They also satisfy the equations (17)

$$\dot{s}_1 = \frac{\sqrt{f(s_1)\varphi(s_1)}}{s_1 - s_2}, \quad \dot{s}_2 = -\frac{\sqrt{f(s_2)\varphi(s_2)}}{s_1 - s_2}, \tag{27}$$

where

$$\begin{aligned} f(s) &= \left(2s + \frac{1}{4}(2h - x) + \frac{1}{4}k\right)\left(2s + \frac{1}{4}(2h - x) - \frac{1}{4}k\right), \\ \varphi(s) &= 4s^3 + 2hs^2 + \left(\frac{1}{16}(2h - x)^2 - \frac{k^2}{16} + \frac{c}{4}\right)s + \frac{l^2}{16}. \end{aligned} \tag{28}$$

(The variable s_1 varies from 0 to ∞, and s_2 parameterizes a circle.)

Let us extract a motion integral $\varkappa = \frac{1}{16}\left((2h - x)^2 - k^2\right)$ in (28):

$$\begin{aligned} f(s) &= 4s^2 + (2h - x)s + \varkappa, \\ \varphi(s) &= s(4s^2 + 2hs + \varkappa + \frac{c}{4}) + \frac{l^2}{16} \end{aligned} \tag{29}$$

and exclude the variable $\varkappa$ from (29). The obtained equation express the energy h as a function of s_i and $\dot{s}_i$:

$$\begin{array}{c} h = -2(s_1 + s_2) + \frac{l^2}{64 s_1 s_2} + \frac{x}{4} + \frac{\sqrt{a_1^2 + x_1^2} - \sqrt{a_2^2 + x_2^2}}{s_1 - s_2}, \\ a_i = \frac{16 s_i^2 x + 4 s_i c + l^2}{64 s_i}, \quad x_i = \frac{(s_1 - s_2)\dot{s}_i}{2\sqrt{s_i}}. \end{array} \tag{30}$$

Let us introduce the conjugated momenta p_i instead of the velocities $\dot{s}_i$:

$$p_i = \int \frac{\partial h}{\partial \dot{s}_i} \frac{d\dot{s}_i}{\dot{s}_i} + F(s_i), \tag{31}$$

where $F(s_i)$ is an arbitrary function of s_i. Note that the addition of $F(s_i)$ does not change the motion equations (the canonical transformation).

We obtain

$$p_i = \frac{1}{2\sqrt{s_i}} \ln \frac{x_i + \sqrt{x_i^2 + a_i^2}}{a_i}. \tag{32}$$

Let us write down the Hamiltonian of the Kovalevskaya system (30) in the variables s_i, p_i

$$h = -s_1 - s_2 + \frac{l^2}{8s_1s_2} + \frac{x}{8} + \frac{a_1\cosh(2p_1\sqrt{s_1}) - a_2\cos(2p_2\sqrt{-s_2})}{s_1 - s_2}. \tag{33}$$

The variables s_i are separating for the Hamiltonian (33). Introducing the constant of separation $\varkappa_1$, we obtain two equations, which can be integrated separately from each other:

$$\begin{array}{l} 2s_1^2 + s_1\left(h - \frac{x}{4}\right) + \frac{l^2}{64s_1} + \varkappa_1 = a_1\cosh(2p_1\sqrt{s_1}), \\ 2s_2^2 + s_2\left(h - \frac{x}{4}\right) + \frac{l^2}{64s_2} + \varkappa_1 = a_2\cos(2p_2\sqrt{-s_2}). \end{array} \tag{34}$$

Inserting (32) in (34), we obtain the Kovalevskaya equations (27). The separation constant $\varkappa_1$ is connected with the constant $\varkappa$ by the formula $\varkappa = 2\varkappa_1 - \frac{c}{8}$.

Thus, the action in the variables s equals

$$I = \frac{1}{2\pi}\oint p(s)\,ds,$$

i. e. is proportional to the area, restricted by a phase curve. The integration region depends on the parameters and values of integrals of the system.

2 The Goryachev-Chaplygin Case (1903)

2.1 Integrals and Separation of Variables

The body is dynamically symmetric in this case: $I_1 = I_2 = 4I_3$, the center of mass is situated in the equatorial plane of the inertia ellipsoid. The Hamiltonian and the additional integral have the form

$$\begin{aligned} H &= \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) + \gamma_1, \\ F_3 &= M_3(M_1^2 + M_2^2) + M_1\boldsymbol{\gamma}_3. \end{aligned} \tag{35}$$

The gyrostat in the Goryachev-Chaplygin case was introduced by L. N. Sretensky (1963):

$$\begin{aligned} H &= \frac{1}{2}\left(M_1^2 + M_2^2 + 4\left(M_3 - \frac{k}{4}\right)^2\right) + r_1\gamma_1, \\ F &= (2M_3 - k)(M_1^2 + M_2^2) - 2r_1M_1\gamma_3. \end{aligned}$$

Let us note that the gyrostatic moment is directed along the axis of dynamical symmetry in generalizations of the Kovalevskaya and Goryachev-Chaplygin cases. A separation of variables in the Sretensky case (a generalization of the Goryachev-Chaplygin case) is described in [19].

A generalization of the Goryachev-Chaplygin case at the zero area constant was suggested in the work of D. N. Goryachev [20]. The generalized Hamiltonian and the integral have the form

$$\begin{gathered} H = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) - r\gamma_1 + \frac{a}{\gamma_3^2}, \\ F = M_3\left(M_1^2 + M_2^2 + 2\frac{a}{\gamma_3^2}\right) + r\gamma_3 M_1, \\ a,\ r = const. \end{gathered} \tag{36}$$

I. V. Komarov and V. B. Kuznetsov in [21] added a constant gyrostatic moment to this Hamiltonian (also at $(\boldsymbol{M},\, \boldsymbol{\gamma}) = 0$) analogous to the Sretensky generalization

$$\begin{gathered} H = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) - r\gamma_1 + \lambda M_3 + \frac{a}{\gamma_3^2}, \\ F = \left(M_3 + \frac{\lambda}{2}\right)\left(M_1^2 + M_2^2 + 2\frac{a}{\gamma_3^2}\right) + r\gamma_3 M_1, \\ a,\ r,\ \lambda = const. \end{gathered} \tag{37}$$

We shall give an analog of the Andoyer-Deprit variables for the bundle of brackets $\mathcal{L}_x$. Let us use of the following sequent of embeddings $\mathbb{R}^1 \subset SO(3) \subset \mathcal{L}_x$.

Let M_3 be equal to one of the momentum variables

$$M_3 = L. \tag{38}$$

Its canonically conjugated coordinate $l(\{l,\, L\} = 1)$ on $so(3)$ can be found by integration of the Hamiltonian flow with the Hamiltonian $\mathcal{H} = L$

$$\frac{dM_1}{dl} = \{M_1,\, L\} = M_2, \quad \frac{dM_2}{dl} = \{M_2,\, L\} = -M_1, \quad \frac{dM_3}{d\tau} = \{M_3,\, L\} = 0. \tag{39}$$

Further taking into account the commutation relation $\{M_2,\, M_2\} = -M_3$ we obtain

$$M_1 = \sqrt{G^2 - L^2}\sin l, \quad M_2 = \sqrt{G^2 - L^2}\cos l, \tag{40}$$

where $G^2 = M_1^2 + M_2^2 + M_3^2$ is a Casimir function of the subalgebra $so(3)$. Let us choose G as the other momentum variable. Then the corresponding flow on the total algebra $\mathcal{L}_x$ has the form

$$\frac{d\boldsymbol{M}}{dg} = 0, \quad \frac{d\boldsymbol{p}}{dg} = \frac{1}{G}\boldsymbol{p} \times \boldsymbol{M}, \tag{41}$$

where g is the variable canonically conjugated to G.

Remark. *The choice of G as a new canonical variable (not G^2) is determined by the fact that the corresponding variable g varies in the range from 0 to 2π and does not depend on the value of G.*

Accordingly to (41), $\boldsymbol{M}$ does not depend on g. Using (41) for $\boldsymbol{p}$ and the Casimir functions

$$(\boldsymbol{M}, \boldsymbol{p}) = H, \quad p^2 + xM^2 = c$$

we obtain

$$\begin{gathered}\boldsymbol{p} = \frac{H}{G^2}\boldsymbol{M} + \frac{\alpha}{G}(\boldsymbol{M} \times \boldsymbol{e}_3 \sin g + G\boldsymbol{M} \times (\boldsymbol{M} \times \boldsymbol{e}_3)\cos g), \\ \alpha^2 = \frac{c - xG^2 - \frac{H^2}{G^2}}{G^2 - L^2}, \quad \boldsymbol{e}_3 = (0, 0, 1).\end{gathered} \tag{42}$$

Thus, (38), (40), (42) define symplectic coordinates on the entire bundle $\mathcal{L}_x$. These coordinates at $x = 0$ pass into known Andoyer-Deprit variables in rigid body dynamics.

We shall find a generalization of the Goryachev-Chaplygin case on $\mathcal{L}_x$ using (38), (40), (42). Let us take the Hamiltonian in the form

$$\mathcal{H} = \frac{1}{2}(G^2 + 3L^2) + \lambda L + a\big(\cos l \cos g + \frac{L}{G}\sin l \sin g\big), \tag{43}$$

where a, λ are constants.

In comparison with [17], a linear with respect to L term describing the gyrostat is added in (43).

The system (43) admits a separation of variables. Indeed, let us perform a canonical change of variables

$$L = p_1 + p_2, \quad G = p_1 - p_2, \quad q_1 = l + g, \quad q_2 = l - g. \tag{44}$$

The Hamiltonian (43) can be represented as

$$\mathcal{H} = \frac{1}{2}\frac{p_1^3 - p_2^3}{p_1 - p_2} - \lambda\frac{p_1^2 - p_2^2}{p_1 - p_2} + \frac{a}{p_1 - p_2}(p_1 \sin q_1 + p_2 \sin q_2). \tag{45}$$

Let us express the Hamiltonian (43) via the variables $\boldsymbol{M}$, $\boldsymbol{p}$ at the zero area constant

$$(\boldsymbol{M}, \boldsymbol{p}) = H = 0.$$

We obtain

$$\mathcal{H} = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) - \lambda M_3 + a\frac{p_1}{|\boldsymbol{p}|}.$$

The additional integral in this case has the form

$$\mathcal{H} = (M_3 - \frac{\lambda}{2})(M_1^2 + M_2^2) - aM_1\frac{p_3}{|\boldsymbol{p}|}.$$

For the algebra $e(3)$ $|\boldsymbol{p}| = 1$, thus we obtain the Goryachev-Chaplygin case.

2.2 The Action-Angle Variables for the Goryachev-Chaplygin Case

The variables q_1, q_2 (44) are already separable. Moreover, they commute with each other as it is easy to verify.

Let $\varkappa$ denote the separation constant. Then

$$\begin{aligned} &\frac{p_1^3}{2} - \lambda p_1^2 + a p_1 \sin q_1 - h p_1 = \varkappa, \\ &\frac{p_2^3}{2} - \lambda p_2^2 - a p_2 \sin q_2 - h p_2 = \varkappa. \end{aligned} \tag{46}$$

Remark 2. *Since*

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = -\frac{a p_i \cos q_i}{p_1 - p_2}, \tag{47}$$

using (46) we obtain the Abel equations in the form

$$\begin{gathered} \dot{p}_i = -\frac{\sqrt{\Phi(p_i)}}{2(p_1 - p_2)}, \\ \Phi(z) = 4a^2 z^2 - (2\varkappa - z^3 + 2\lambda z^2 + 2hz)^2. \end{gathered} \tag{48}$$

The action in the variables s equals

$$I = \frac{1}{2\pi} \oint p(s)\, ds.$$

3 The Chaplygin Case

3.1 The Integrals and the Separation of Variables

This case discovered by Chaplygin in 1902 [24] is a particular integrable case at the zero area constant $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$ and a quartic integral. This system is similar to the Kovalevskaya case in the Euler-Poisson equations.

The Hamiltonian and the additional integral of the system have the form [24]

$$\begin{aligned} H &= \frac{a}{2}(M_1^2 + M_2^2 + 2M_3^2) + \frac{c}{2}(\gamma_1^2 - \gamma_2^2), \\ F &= (M_1^2 - M_2^2 + \frac{c}{a}\boldsymbol{\gamma}_3^2)^2 + 4M_1^2 M_2^2. \end{aligned} \tag{49}$$

Let us consider the integration of the Kirchhoff equations using the method of separation of variables in the Chaplygin case on the bundle of brackets $\mathcal{L}_x$ (the integration in this case has not been indicated before). The separation of

variables in this problem on was performed by S. A. Chaplygin [24] $e(3)$, and by O. I. Bogoyavlensky [12] on $so(4)$.

The Hamiltonian and the integrals on the bundle have the form

$$H = \tfrac{1}{2}(\alpha_2 M_1^2 + \alpha_1 M_2^2 + (\alpha_1 + \alpha_2) M_3^2 - (a_1 - a_2)(\gamma_1^2 - \gamma_2^2)) = \tfrac{h}{2},$$
$$J_2 = x(M_1^2 + M_2^2 + M_3^2) + \gamma_1^2 + \gamma_2^2 + \gamma_3^2,$$
$$J_3 = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3 = 0,$$
$$J_4 = (\alpha_1 M_1^2 - \alpha_2 M_2^2 - (a_1 - a_2)\gamma_3^2)^2 + 4\alpha_1\alpha_2 M_1^2 M_2^2 = k^2,$$

where $\alpha_1 = 1 - xa_1$, $\alpha_2 = 1 - xa_2$ $(k > 0)$. Let us introduce the separating variables s_1 and s_2 by formulae [12, 24]

$$\begin{gathered} s_1 = \tfrac{u+k}{v}, \quad s_2 = \tfrac{u-k}{v}, \\ u = \alpha_1 M_1^2 + \alpha_2 M_2^2, \quad v = (a_2 - a_1)\gamma_3^2. \end{gathered} \tag{50}$$

The evolution of s_1 and s_2 is determined by degenerate Abel-Jacobi equations:

$$\dot{s}_1 = -\sqrt{(1 - s_1^2)(\delta_1 - \beta_1 s_1)}, \quad \dot{s}_2 = -\sqrt{(1 - s_2^2)(\delta_2 - \beta_2 s_2)}, \tag{51}$$

where

$$\delta_1 = m_1(h + k) - m_2 J_2, \quad \delta_2 = m_1(h - k) - m_2 J_2,$$
$$\beta_1 = x(a_1 - a_2)(h + k) - m_3 J_2, \quad \beta_2 = x(a_1 - a_2)(h - k) - m_3 J_2,$$
$$m_1 = \alpha_1 + \alpha_2, \quad m_2 = x(a_1 - a_2)^2, \quad m_3 = (a_1 - a_2)(\alpha_1 + \alpha_2).$$

Thus, the motion equations can be integrated by means of elliptic functions of time.

3.2 The Action-Angle Variables for the Chaplygin System on the Bundle of Brackets

The variables s_1, s_2 (50) commute at the zero value of the integral $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$.

We find the energy $E = 2h$ as a function of s_1, s_2, $\dot{s}_1$, $\dot{s}_2$ from (51):

$$\begin{aligned} E = \frac{\dot{s_1}^2}{(1 - s_1^2)(s_1 a_{12} + m_1)} + \frac{\dot{s_2}^2}{(1 - s_2^2)(s_2 a_{12} + m_1)} - \frac{2J_2 m_3}{a_{12}} + \\ + \frac{J_2(m_2 a_{12} + m_1 m_3)}{a_{12}} \left(\frac{1}{s_1 a_{12} + m_1} + \frac{1}{s_2 a_{12} + m_1} \right), \end{aligned} \tag{52}$$

where $a_{12} = x(a_2 - a_1)$. Introducing the conjugated momenta (31)

$$p_i = \frac{2\dot{s}_i}{(1 - s_i^2)(s_i a_{12} + m_1)},$$

we obtain the Hamiltonian in separated variables

$$
\begin{aligned}
H = \frac{1}{4}(1-s_1^2)(s_1a_{12}+m_1)p_1^2 + \frac{1}{4}(1-s_2^2)(s_2a_{12}+m_1)p_2^2 - \frac{2J_2m_3}{a_{12}} + \\
+ \frac{J_2(m_2a_{12}+m_1m_3)}{a_{12}}\left(\frac{1}{s_1a_{12}+m_1} + \frac{1}{s_2a_{12}+m_1}\right).
\end{aligned}
\tag{53}
$$

As it may be expected, the variables are separated in the Hamiltonian (53):

$$
\begin{aligned}
&\frac{1}{4}(1-s_1^2)(s_1a_{12}+m_1)p_1^2 + \frac{J_2(m_2a_{12}+m_1m_3)}{a_{12}(s_1a_{12}+m_1)} = \varkappa, \\
&\frac{1}{4}(1-s_2^2)(s_2a_{12}+m_1)p_2^2 + \frac{J_2(m_2a_{12}+m_1m_3)}{a_{12}(s_2a_{12}+m_1)} = h + \frac{2J_2m_3}{a_{12}} - \varkappa.
\end{aligned}
\tag{54}
$$

The action in s equals

$$
I = \frac{1}{2\pi}\oint p(s)\,ds.
$$

4 The Bogoyavlensky System

The particular Bogoyavlensky case [12] with the Hamiltonian

$$
H = \frac{1}{2}\left(a_1M_1^2 + a_2M_2^2 + a_3M_3^2 + \frac{1}{2x}((a_2+a_3)\gamma_1^2 + (a_1+a_3)\gamma_2^2 + (a_1+a_2)\gamma_3^2)\right) = \frac{h}{2}
$$

can be integrated by means of elliptic functions. The system possesses the integrals

$$
\begin{aligned}
&J_2 = x(M_1^2+M_2^2+M_3^2) + \gamma_1^2+\gamma_2^2+\gamma_3^2, \\
&J_3 = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3 = 0, \\
&J_4 = ((a_3-a_2)\gamma_1^2 + (a_1-a_3)\gamma_2^2 + (a_1-a_2)\gamma_3^2)^2 + 4(a_3-a_1)(a_3-a_2)\gamma_1^2\gamma_2^2 = h^2;
\end{aligned}
$$

On $so(4)$ the integration method is developed in [12]. If we introduce the variables

$$
s_1 = \frac{u+h}{v}, \quad s_2 = \frac{u-h}{v},
$$

where $u = (a_3-a_2)\gamma_1^2 + (a_3-a_1)\gamma_2^2$, $v = (a_1-a_2)\gamma_3^2$, the motion equations can be reduced to the form

$$
\dot{s}_1 = -\sqrt{(1-s_1^2)(\delta_1-\beta_1s_1)/2}, \quad \dot{s}_2 = -\sqrt{(1-s_2^2)(\delta_2-\beta_2s_2)/2}, \tag{55}
$$

and the constants δ_i, β_i are expressed as follows:

$$
\begin{gathered}
\delta_1 = (J_1+h/2)n_1 + J_2n_2, \quad \delta_2 = (J_1-h/2)n_1 + J_2n_2, \\
\beta_1 = (a_1-a_2)(J_2a_3 - J_1 - h/2), \quad \beta_2 = (a_1-a_2)(J_2a_3 - J_1 + h/2), \\
n_1 = 2a_3 - a_1 - a_2, \quad n_2 = 2a_1a_2 - a_3(a_1+a_2).
\end{gathered}
$$

Remark 3. *Let us note that (51, 55) are degenerate Abel-Jacobi equations, i. e. each of them depends on a unique variable* s_1 *or* s_2 *only, and a two-dimensional Abel torus splits into one-dimensional tori.*

5 Integrable Systems on a Sphere with a Cubic Integral (the Gaffet System)

In the B. Gaffet work [13] an integrable case on the two-dimensional sphere $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$ with the potential $U = a(\gamma_1\gamma_2\gamma_3)^{-2/3}$ is found.

This integrable system on $e(3)$ and at the zero value of the area integral possesses the integrals

$$\begin{aligned} H &= \frac{1}{2}(M_1^2 + M_2^2 + M_3^2) - \frac{1}{2}\frac{a^2(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)}{(\gamma_1\gamma_2\gamma_3)^{2/3}}, \\ J &= M_1M_2M_3 + a^2\left(\frac{M_1}{\gamma_1} + \frac{M_2}{\gamma_2} + \frac{M_3}{\gamma_3}\right)(\gamma_1\gamma_2\gamma_3)^{1/3}. \end{aligned} \tag{56}$$

Let us note that the integrability of the system and the form of integrals (56) are preserved on the total bundle of brackets $\mathcal{L}_x$ and at the zero value $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$.

Despite some particular results [14, 13], an explicit integration of (56) has not been performed yet. $L - A$ pair for this system is indicated in [15] and has the form:

$$\frac{d}{dt}\mathbf{L} = [\mathbf{L}, \mathbf{A}],$$

$$\mathbf{L} = \begin{pmatrix} \lambda & M_3 + ay_3 & M_2 - ay_2 \\ M_3 - ay_3 & \lambda & M_1 + ay_1 \\ M_2 + ay_2 & M_1 - ay_1 & \lambda \end{pmatrix},$$

$$\mathbf{A} = \tfrac{2a}{3}(\boldsymbol{y}, \boldsymbol{y}) \begin{pmatrix} 0 & y_3^{-1} & y_2^{-1} \\ y_3^{-1} & 0 & y_1^{-1} \\ y_2^{-1} & y_1^{-1} & 0 \end{pmatrix},$$

where $\boldsymbol{y} = \frac{\gamma}{(\gamma_1\gamma_2\gamma_3)^{1/3}}$.

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